The Perpetual Variance Swap

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Abstract

Variance Swap is a constant function market maker (CFMM) with its trading function φ derived from a replicated logarithmic payoff. While the payoff is concave, future applications include replicating the payoff of a variance swap for a liquidity provider to hedge against volatility of the underlying assets or offer constant gamma exposure. This can be done by shorting LP shares or auctioning off the expected arbitrage payoff.

1 Introduction

Constant Function Market Makers (CFMMs) are a subclass of Automated Market Makers (AMM) that utilize a trading function φ to maintain a desired ratio of reserves $R \in \boldsymbol{R}_+{}^n$ held by the AMM. This creates a generalized pricing algorithm, represented as a curve that multiple agents pool liquidity to. Thereby, a liquidity provider is in effect holding a portfolio of tokens and exposed to the volatility risks of the underliers. More importantly, the invariant quotes a price to an agent for some scalar quantity in a two-asset trade. This proposal (Δ, Λ) is accepted by the AMM if the trading function $\varphi(R)$ is unchanged, the fee γ is < 1, and the amount of reserves R is sufficient as described by

$$\varphi(R + \gamma\Delta - \Lambda) = \varphi(R) \tag{1}$$

Due to the storage constraints of a blockchain, the high fees per transaction have propelled constant function AMMs as the go-to mechanism for on-chain decentralized exchanges and derivative exposure. Uniquely, AMMs can create liquid markets for derivative products independent of sophisticated market makers.

1.1 Replicating Market Maker

A replicating market maker (RMM) is a CFMM that is parameterized by a portfolio value function $V : R^n_+ \to R$. Given a constant k, that ψ is a non-decreasing function, and our CFMM is path independent we can define our Portfolio Value Function V to be:

$$V(p) = \inf\{p^T R|\psi(R) \geq k, R \in R^n\} \tag{2}$$

where $k \in R$ is some constant, R is the set of our reserve asset quantities, and $p \in R^n$ is the associated price vector of our reserve set after arbitrage. Since the discrete space of particular concave payoff function and the space of a Constant Function Market Maker (CFMM) are equivalent by conjecture, the portfolio value function of a very CFMM has an replicatable payoff matched to $V(p)$. Making it possible to construct trading functions from a desired payoff and vice versa.

The implication of RMMs for the highly lucrative derivative market is substantial since RMMs can replicate the payoff of derivative instruments with concave or bounded convex payoffs without a volatility oracle or. the need for counterparties. Instead the CFMM itself becomes the counterparty for each trade-bootstrapping liquidity and the LP share can replicate a specific derivative payoff.

2 Market Model

Consider a RMM with a numeraire, USDC and and a risky asset, ETH. Let t and r be the current reserves of USDC and ETH tokens respectively.

To find the token exchange price is, we determine the ratio of t and r so that the invariant $t + \ln(r) = k$ is preserved. Thereby when you sell Δr tokens, you will get Δt tokens such that $t + \ln(r) = (t + \Delta t) + \ln(r + \Delta r)$. The same holds true when you sell Δt, giving you the relation such that $t + \ln(r) = (t + \Delta t) + \ln(r + \Delta r)$.

2.1 Trading Function

The trading function is $\psi(R_1, R_2) = R_1 + log(p_0 R_2)$

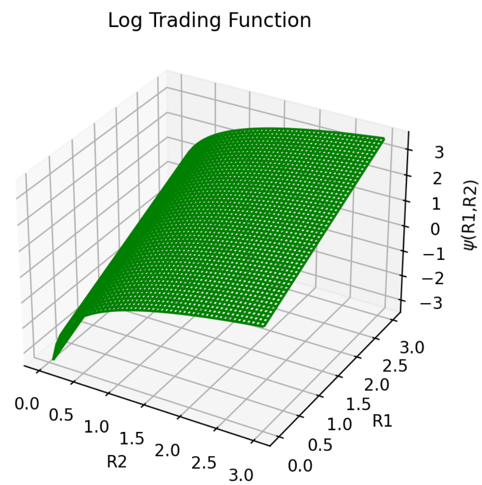

Figure 1: Graph of Trading Function

2.2 Trading Δt with Δr

The swap calculation as it relates to the amount of two reserves before and after a swap is demonstrated in this section.

2.2.1 Deriving t' and Δt

This is the situation where Δr is known, meaning that an agent knows how many units of the risky asset they are willing to buy or sell.

Let γ represent $1 - p$ with p being the fee.
Let α represent $\frac{\Delta r}{r}$.
Let t' equal $t + \Delta t$, our total reserves of USDC after a swap

$$(t + \Delta t) + \ln(\gamma(r + \Delta r)) = t + \ln(r) \tag{3}$$

$$
\begin{aligned}
t' + \ln(\gamma(r+\Delta r)) &= t + \ln(r) \\
t' &= t + \ln(r) - \ln(\gamma(r+\Delta r)) \\
&= t + \ln(\frac{r}{\gamma(r+\Delta r)}) \\
&= t + \ln(\frac{1}{\gamma(1+\frac{\Delta r}{r}})) \\
&= t + \ln(\frac{1}{\gamma(1+\alpha)})
\end{aligned}
$$

Final Equations for t' and Δt:

$$t' = t + \ln(\frac{1}{\gamma(1+\alpha)}) \tag{4}$$

$$\Delta t = \ln(\frac{1}{\gamma(1+\alpha)}) \tag{5}$$

2.2.2 Deriving r' and Δr

This is the situation where Δt is known, meaning that an agent knows how many units of the stable asset they are willing to buy or sell.

Let γ represent $1-p$ with p being the fee.
Let r' equal $r+\Delta r$, our total reserves of ETH after a swap

$$(t+\Delta t) + \ln(\gamma(r+\Delta r)) = t + \ln(r) \tag{6}$$

$$
\begin{aligned}
\Delta t + \ln(\gamma(r')) &= \ln(r) \\
\ln(\gamma(r')) &= \ln(r) - \Delta t \\
\gamma(r') &= e^{\ln(r)-\Delta t} \\
\gamma(r') &= \frac{r}{\exp(\Delta t)} \\
r' &= \frac{r}{\gamma \exp(\Delta t)}
\end{aligned}
$$

Final Equations for r' and Δr:

$$r' = \frac{r}{\gamma \exp(\Delta t)} \tag{7}$$

$$\Delta r = \frac{r}{\gamma \exp(\Delta t)} - r \tag{8}$$

2.3 Liquidity Provider Payoff

The payoff of the Liquidity Provider is a concave curve that is a function of the Log curve. The payoff is logarithmic and based off the trading function $\varphi(p) = x + log(y) = k$. Derived in *Replicating Monotonic Payoffs Without Oracles* the payoff is:

$$f(p) = \begin{cases} 0, & \text{if } i < k \\ \ln(p/p_0), & \text{if } i \geq k \end{cases} \tag{9}$$

2.4 Expected Arbitrageur Payoff

Our Arbitrageur Payoff replicates the Payoff of a variance swap without the use of an oracle. Derived by Angeris et al. (2021) in *Replicating Monotonic Payoffs Without Oracles*, our expected arbitrage payoff is a polynomial function on the annualised volatility of ETH. The Arbitrageur Payoff is the Expected Value of the Geometric Brownian Motion.

Let σ represent the volatility implied in the price of contract.
Let K_{vol} represent the outcome/strike volatility of ETH.
Let T equal the total trade size.
Our equation for the expected arbitrage payoff is:

$$\text{Arbitrageur Payoff} = \frac{1}{2}(\sigma^2 - K_{vol}^2)T \tag{10}$$

This formula for the Arbitrageur Payoff replicates the payoff of a variance swap $N_{var}(\sigma^2 - K_{vol}^2)$ derived in Rouah's *Variance Swap* with N_{var} being the Notional Variance. Since Variance Swap's payoff is approximately the same as the Variance Payoff, performing Swaps on Variance Swap's protocol allows the Arbitrageur to replicate variance swaps and get a payoff related to the volatility of ETH.

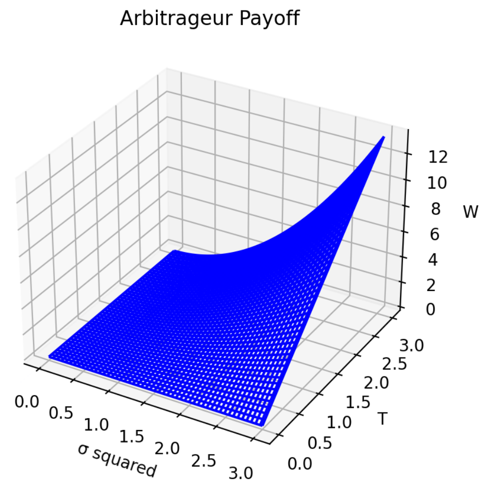

Figure 2: Graph of Arbitrage Payoff

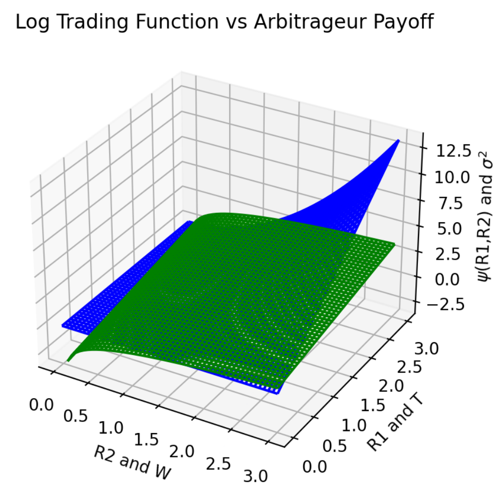

Figure 3: Graph of Trading Function With Arbitrage Payoff

3 Future Work

3.1 Growth Rate of LP

We aim to first compute the geometric mean as the LP share growth rate is compounded. Following a Geometric Brownian Motion we find variance, σ^2

$$\sigma^2 = \frac{\log(E[PV]E[k\sqrt{m_p^T}])}{-T}$$

3.2 Constructing a Convex Payoff

A liquidity provider's portfolio value is $V(p) = f(p) + p \cdot g(p)$, where p is the price of the risky asset in terms of the stable asset, $f(p)$ is the units held of the stable asset at price p, and $g(p)$ is the units held of the risky asset at price p ($g(p)$ is also called the replication cost).

To replicate the logarithmic payoff, the liquidity provider should hold the stable and risky assets at these amounts at any price p:

$$f(p) = \begin{cases} 0, & \text{if } p < p_0 \\ \ln(p/p_0), & \text{if } p \geq p_0 \end{cases} \quad (11)$$

$$g(p) = \begin{cases} 1/p_0, & \text{if } p < p_0 \\ 1/p_0, & \text{if } p \geq p_0 \end{cases} \quad (12)$$

Given the payoff of the portfolio value function, $V(p) = log(p/p_0) + p * 1/p = log(p/p_0) + 1$, which is logarithmic and hence strictly concave, a LP share holder would need to short the LP share to yield an inverse payoff which would be convex. Note this is the portfolio value function when $p \geq p_0$.

Natural convexity allows a risk adjusting investor earn on the underlying changes in prices. Offering a potential hedge to impermanent loss.

3.3 Hedging Against Impermanent Loss

Shorting a LP share of Variance Swap allows the creation of a convex payoff that can be used to hedge against the impermanent loss caused by sharp volatility spikes in CFMMs using the $xy = k$ curve.

Let L represent the Virtual Liquidity in the pool.
Let p_a and p_b be the left and right price bounds of the asset respectively
Let p be the current price of the asset

The Replicating Portfolio of a Constant Product Market with Bounded Liquidity derives that the liquidity provider's payoff for Uniswap V3 is:

$$U(p) \; = 2L\sqrt{p} - L\sqrt{p_a} - \frac{pL}{\sqrt{p_b}} \tag{13}$$

In order to hedge against impermanent loss, we want to perform a Delta-Gamma hedge on the CFMM payoff curve. A Delta-Gamma hedge is a bet that volatility will rise or decline sharply and by performing a successful delta and gamma hedge, the Logarthmic Payoff will successfully hedge against the impermanent loss caused by large shifts in volatility of the underlying asset. We first want to find the Delta (1st derivative) and Gamma (2nd Derivative) of the CFMM payoff.

$$U'(p) = L\left(\frac{1}{\sqrt{p}} - \frac{1}{\sqrt{p_b}}\right) \tag{14}$$

$$U''(p) = -0.5Lp^{-\frac{3}{2}} \tag{15}$$

After shorting the payoff of the logarithmic curve, we get a $S(p) = -V(p)$ as our new payoff, resulting in $S(p) = -log(p/p_0) - 1$.

$$S'(p) = -\frac{1}{p} \tag{16}$$

$$S''(p) = \frac{1}{p^2} \tag{17}$$

Dividing Variance Swap's Gamma from the CFMMs Gamma gets our hedge.

$$H(p) = \frac{U''(p)}{S''(p)} = \frac{0.5Lp^{-\frac{3}{2}}}{\frac{1}{p^2}} \tag{18}$$

In order to accomplish a hedge a concave LP share, we will want to construct a general convex position and lend out the position. In essence you are splitting the LP position in two where the one side is concave and the other is bounded convex to replicate a power payoff.

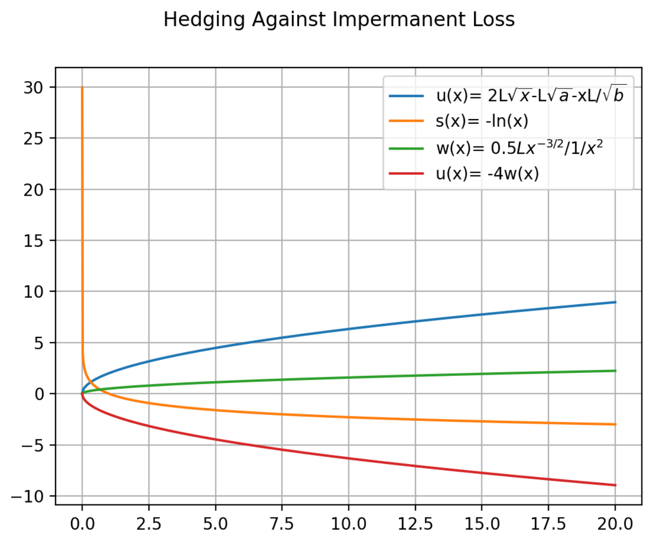

Figure 4: Graph of Hedging Against Impermanent Loss

We can show that this successfully hedges against impermanent loss by subtracting the hedge from our payoff, giving us a Gamma-Neutral position.

Due to Uniswap-V3 using liquidity bounds, Gamma hedging may not be enough to successfully hedge against Impermanent Loss. If Gamma hedging is not sufficient, further hedging can be performed by buying or selling ETH futures to get a Delta-Neutral position.

References

[1] Fabrice Rouah. "Variance Swap". *frouah*

[2] Guillermo Angeris, Tarun Chitra, Alex Evans. "Replicating Monotonic Payoffs Without Oracles". *arxiv.org/abs/2111.13740*, 2021

[3] Neuberger, A. "The log contract". In *Journal of Portfolio Management*, 1994

[4] Guillermo Angeris, Alex Evans, Tarun Chitra. "Replicating Market Makers". *https://arxiv.org/abs/2103.14769*, 2021